Highways for a circular city

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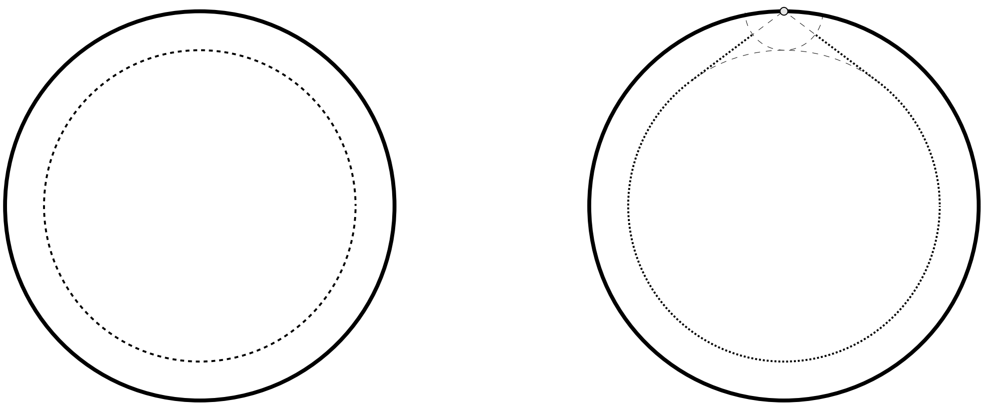
At our meeting, Naoki Katoh proposed the following problem. Given a city C that is for the beginning assumed to be a circle of radius R. What is the best way to build a highway H of given length l, to minimize the maximum travel time between any two points of C, if we assume that traveling on the highway is instantaneous, so the travel time is only the distance not travelled by the highway. For any two points $p, q \in C$ we have the choice of going directly from p to q, or going from p to the nearest point on H, then traveling on the highway to the nearest point to q, and leaving the highway there. So the travel distance is $\min(d(p,q), d(p,H)+d(H,q))$, where, as usual, the distance from a point to a set denotes the distance to the nearest point of that set. We want to choose H in such a way that the total length of H is l and $\max\limits_{p,q \in C} \min(d(p,q), d(p,H)+d(H,q))$ is as small as possible. It turns out to be easier to look at the equivalent problem of giving the maximum travel distance δ and asking for the minimum length of H with $\min(d(p,q), d(p,H)+d(H,q)) \leq \delta$ for all $p, q \in C$. The advantage is that this is an isoperimetric-type problem for the curve H, so H must locally either minimize length, that is, be straight, or be subject to some restriction such that locally straightening it would increase the maximum travel distance.

Depending on our assumptions on the highway H, we obtain at least two distinct problems, when H is an open or closed curve. The closed curve situation is easier. I made the claim that for the closed curve the solution is always the inner parallel curve of C at distance $\frac{1}{2}\delta$, if it exists, so if C is a circle of radius R, the H is a circle of radius $R - \frac{1}{2}\delta$. And for the open curve, the solution consists of a part of that inner parallel curve, with two line segments at beginning and end, which are tangential to the curve and whose continuations intersect in a point of C whose distance to beginning and end is $\frac{1}{2}\delta$.



EXTREMAL OPEN AND CLOSED HIGHWAYS (right: conjectured...)

Unfortunately, my handwaving argument does not really work. Instead I can show here a simple analytic proof for the extremality of the circle as closed highway, but again it does have a limitation, it works only for δ not too large, e.g., $\delta \leq \frac{3}{2}R$. We need some properties on the structure of the nearest-neighor map ν from C to H. These do not depend on C being a circle.

- Nearest-neighbor assignments never cross: if $p, q \in C$, then the segments $p\nu(p)$ and

$q\nu(q)$ do not intersect. This follows from the triangle inequality, and implies that ν is a monotone function from C to H.

- H is a convex curve: if H were somewhere concave, we could decrease the length by making it straight, and would not increase the distance to the nearest neighbor for any point of C.
- each point of C has a unique nearest neighbor on H, at least if H is closed. This follows from the previous property, the nearest-neighbour mapping to a convex set is always unique. It is also continuous. Together with monotone and not constant, this implies surjective: each point of H is nearest neighbor for some point of C.

With this, we can now write down the proof for the closed highway. We parametrize the highway by its polar coordinates with respect to the center of C; let $r(\varphi)$ be the radial distance of the point at angle φ. The length of a curve given in polar coordinates is $\int_0^{2\pi}\sqrt{(r(\varphi))^2+(r'(\varphi))^2}d\varphi$. Let $p(\varphi)$ be the point on C that has the point at angle φ as its nearest neighbor. This distance is at least $R-r(\varphi)$. Consider now the points $p(\varphi)$ and $p(\varphi+\pi)$. Suppose their distance $d(p(\varphi),p(\varphi+\pi))$ is larger than δ, then we have $d(p(\varphi),H)+d(H,p(\varphi+\pi))\leq\delta$. Since $d(p(\varphi),H)\geq R-r(\varphi)$, we can then use $2R-r(\varphi)-r(\varphi+\pi)\leq\delta$, so $r(\varphi)+r(\varphi+\pi)\geq 2R-\delta$, and

$$\begin{aligned}\text{length}(H) &= \int_0^{2\pi}\sqrt{(r(\varphi))^2+(r'(\varphi))^2}d\varphi \\ &\geq \int_0^{2\pi}(r(\varphi))d\varphi = \int_0^{\pi}(r(\varphi))+r(\varphi+\pi)d\varphi \geq \int_0^{\pi}2R-\delta d\varphi \\ &= 2\pi(R-\frac{1}{2}\delta),\end{aligned}$$

with equality only if the radius $r(\varphi)$ is constant.

Bur it remains to show that the points $p(\varphi)$ and $p(\varphi+\pi)$ have distance larger than δ. Since H is convex, it is easy to see that $d(p(\varphi),p(\varphi+\pi))\geq r(\varphi)+r(\varphi+\pi)$. Since for any point p on C there is an antipodal point $-p$ whose distance is $2R>\delta$, we have $d(p,H)+d(-p,H)\leq\delta$, so $R-r(\varphi)\leq d(p(\varphi),H)\leq\delta$, thus $r(\varphi)\geq R-\delta$. Therefore $d(p(\varphi),p(\varphi+\pi))\geq 2R-2\delta$, which is at least δ if $\delta<\frac{3}{2}R$.

This is, of course, not really satisfactory. I hope someone else has a better idea.