Casting an Object with a Core*

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Abstract

This paper addresses geometric problems in manufacturing objects by casting. In casting, molten material is poured into the cavity of the cast and allowed to solidify, after which the cast is removed. The cast has two cast parts to be removed in opposite directions. To manufacture more complicated objects, the cast may also have a side core to be removed in a direction skewed to the removal directions for the cast parts. We address the following problem: Given an object and the removal directions for the cast parts and the side core, can a cast be constructed such that the cast parts and the side core can be removed in the directions specified without colliding with the object or each other? We give necessary and sufficient conditions for the problem, as well as a discrete algorithm to perform the test in $O(n^3 \log n)$ time for polyhedral objects, where n is the number of vertices, edges, and facets. If the test result is positive, a cast with complexity $O(n^3)$ can be constructed within the same time bound. We also present an example to show that a cast may have $\Omega(n^3)$ complexity in the worst case.

1 Introduction

Casting or injection molding [7, 8, 14, 16] is ubiquitous in the manufacturing industry for producing consumer products. A cast can be viewed as a box with a cavity inside. Molten material (such as iron, glass or polymer) is poured into the cavity and allowed to solidify. The cast has two cast parts and it is convenient to view that the hardened object is taken out by removing the two cast parts in opposite directions. In reality, one cast part is fixed and the other cast part retracts, carrying the object with it. Typically, pins slide through tunnels in the retracting cast part and push the object out of the cavity. In the prevailing technology the ejection direction is opposite to the retraction direction [7, 8, 14, 16]. Hence, it is equivalent to view that the two cast parts are removed in opposite directions. This is our assumption throughout this paper unless stated explicitly otherwise. Non-opposite removal directions are allowed in a few previous works and we will state this explicitly in the literature survey.

Many common objects, however, cannot be manufactured using two cast parts alone. Additional pieces, which we refer to as *side cores*, are needed. For example, consider a coffee mug in Figure 1(a). Suppose that we place the coffee mug horizontally as in Figure 1(b). The handle of the mug can be

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produced by the two cast parts designed to be removed vertically in opposite directions. However, these two cast parts cannot produce the cavity of the mug, which needs to be shaped by something that is removed horizontally. Figure 1(b) shows the side core needed to shape the cavity of the mug. Side cores are used widely to enlarge the class of objects manufacturable by casting [6, 14, 16, 17]. Figure 2 shows schematically the removal of the cast parts and the side core.

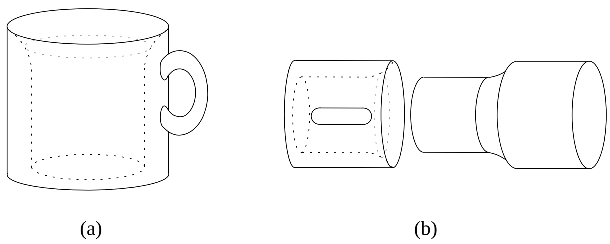


Figure 1: (a) A coffee mug is unattainable using a 2-part cast. (b) By incorporating a side core, the cavity of the coffee mug can be manufactured.

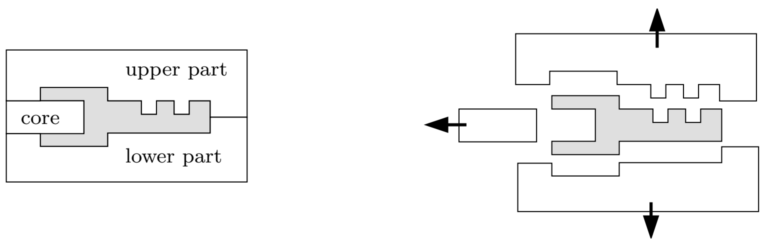


Figure 2: The casting process with a side core is illustrated in 2D.

The casting problem is related to the assembly problem in robotics [9]. A mechanical object is usually made of several components. Thus, it is desirable to automate the assembly process and compute the movements needed to bring the components together. The casting problem is different in that a partition of the cast is sought.

There has been a fair amount of work on the case where no side core is allowed. Given the object and the removal directions for the two cast parts, the object is *castable* if a cast can be constructed such that the cast parts can be removed without being blocked by the object or each other. Kwong [12] reduced the problems of castability testing and cast construction to the hidden surface removal problem. Ahn et al. [4] gave an $O(n \log n)$-time algorithm for castability testing and cast construction for a polyhedron with n vertices. Their approach also leads to an $O(n^4)$-time algorithm for finding all feasible removal directions for the cast parts. Ahn, Cheong and van Oostrum [3] allowed the removal directions to have an uncertainty α for any $\alpha > 0$. Given the removal directions and the uncertainty α, they can test castability and construct the cast in $O(n \log n)$ time. They can also find all feasible removal directions in $O(n^2 \log n/\alpha^2)$ time. Given a convex polyhedron and the removal directions, Majhi, Gupta and Janardan [13] can compute two cast parts that meet at the flattest parting line. Bose et al. [5] studied the case of sand casting in which the partition of the cast into two parts must be done by a plane. They considered both opposite removal directions and non-opposite removal directions for the two cast parts. Ahn, Cheng and Cheong [1] removed the restriction of partitioning using a plane, while allowing non-opposite removal directions. They can test the castability and construct the cast in $O(n^2 \log^2 n)$ time for a given pair of possibly non-opposite removal directions.

In contrast, the algorithmic complexity of allowing for side core(s) is largely unknown. Chen, Chou and Woo [6] described a heuristic to compute the removal directions for the two cast parts that minimizes the number of side cores needed. However, the removal directions returned need not be feasible. Based on the approach of Chen, Chou and Woo, Hui presented exponential time algorithms to construct a cast [10]. However, there is no guarantee that a feasible cast will be found if there is one. Ahn et al. [2] proposed a hull operator, reflex-free hull, to define cavities in polyhedron. The motivation is that the cavities limit the search space for the removal directions for the cast parts and side cores.

In this paper, we study the casting problem when one side core is allowed. Suppose that the object and the removal directions for the two cast parts and the side core are given. We present a necessary and sufficient condition for the object to be castable, assuming that the swept volumes generated by the removal of the cast parts and the side core are required not to intersect the object or each other. For a polyhedron of n vertices, we develop an $O(n^3 \log n)$-time algorithm for performing the castability test. A cast of $O(n^3)$ complexity can be constructed within the same time bound. We give an example to show that a cast of $\Omega(n^3)$ complexity is necessary in the worst case. In the absence of a side core, the swept volumes generated by the removal of the cast parts must not intersect the object or each other, in order that the object can be produced using a 2-part cast [4, 12]. When there is a side core, it is often removed first in practice. So our assumption is stronger than necessary in this case; for example, see Figure 3. Nevertheless, this paper gives, to

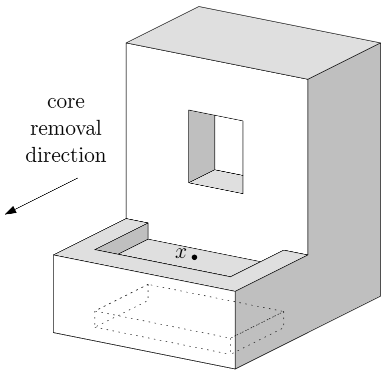


Figure 3: An object with a small hole in the middle of the vertical wall, and bigger holes on the top and bottom face of the horizontal leg. It can be produced using a 2-part cast with a side core if the side core is removed first. Consider a point x in the middle of the hole on the horizontal leg. It must be removed vertically upward, otherwise it intersects the object. But if we remove x vertically upward, we run into the swept volume of the side core, which violates our assumption.

the best of our knowledge, the first characterization of castability and the first polynomial time algorithm for the problem when a side core is allowed.

We provide the basic definitions and notation in Section 2. We prove the characterization of castability in Section 3. In Section 4 we develop the castability testing and cast construction algorithm. We give some experimental results in Section 5. We present a lower bound construction in Section 6 that shows that the cast produced has optimal complexity in the worst-case. We conclude in Section 7.

2 Preliminaries

Let A be a subset of $\mathbb{R}^3$. We say that A is *open* if for any point $x \in A$, A contains some ball centered at x with positive radius. We say that A is *closed* if its complement is open. For all points $x \in \mathbb{R}^3$, x is a *boundary point* of A, if any ball centered at x with positive radius intersects both A and its complement. The *boundary of* A, denoted by $\mathrm{bd}(A)$, is the set of boundary points. The *interior* of A, $\mathrm{int}(A)$, is $A \setminus \mathrm{bd}(A)$. Note that $\mathrm{int}(A)$ must be open.

An object is *monotone* in direction d if for any line ℓ parallel to d, the intersection between ℓ and the object is a single interval.

The *Minkowski sum* of two sets $S_1 \subset \mathbb{R}^3$ and $S_2 \subset \mathbb{R}^3$ is defined as $S_1 \oplus S_2 = \{x + y : x \in S_1, y \in S_2\}$, where x and y are treated as position vectors and $x + y$ is their vector sum.

We assume that the outer shape of the cast equals a box denoted by $\mathcal{B}$. The cavity of $\mathcal{B}$ has the shape of the object $\mathcal{Q}$ to be manufactured. We assume that $\mathcal{Q}$ is an open set so that the cast $\mathcal{B} \setminus \mathcal{Q}$ is a closed set. The box $\mathcal{B}$ is large enough so that $\mathcal{Q}$ is contained strictly in its interior. We use d_p and $-d_p$ to denote the given removal directions for the two cast parts, and d_c to denote the given removal direction for the side core.

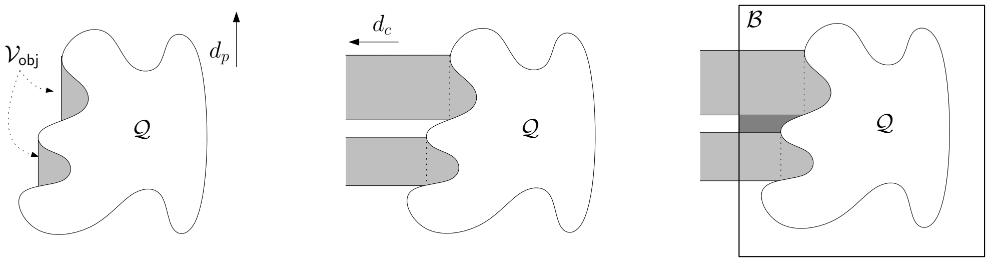
We call the cast part to be removed in direction d_p the *red cast part* and denote it by $\mathcal{C}_r$. We call the other cast part the *blue cast part* and denote it by $\mathcal{C}_b$. We denote the side core by $\mathcal{C}_c$. The cast parts and the side core are required to be connected subsets of $\mathcal{B}$. Together they form the cast, i.e., $\mathcal{B} \setminus \mathcal{Q} = \mathcal{C}_r \cup \mathcal{C}_b \cup \mathcal{C}_c$. The interior of the two cast parts and the side core are mutually disjoint.

Let γ_p and γ_c denote the rays emitting from the origin towards directions d_p and d_c, respectively. Given the object $\mathcal{Q}$ and the directions d_p and d_c, we say that $\mathcal{Q}$ is *castable* if $\mathcal{B} \setminus \mathcal{Q}$ can be partitioned into $\mathcal{C}_r$, $\mathcal{C}_b$ and $\mathcal{C}_c$ such that $\mathcal{C}_r \oplus \gamma_p$, $\mathcal{C}_b \oplus -\gamma_p$, and $\mathcal{C}_c \oplus \gamma_c$ do not intersect $\mathcal{Q}$ or each other.

3 The characterization of castability

In this section we characterize the castability of $\mathcal{Q}$ given the directions d_p and d_c. The object $\mathcal{Q}$ needs not be polyhedral. We first introduce some definitions.

Consider the illumination of $\mathbb{R}^3$ by light sources at infinity in directions d_p and $-d_p$, treating $\mathcal{Q}$ as opaque. We call the subset of non-illuminated points in $\mathcal{B} \setminus \mathcal{Q}$ the *object shadow*. That is, for any point x in the object shadow, the rays emitting from x towards d_p and $-d_p$ intersect $\mathcal{Q}$. We denote the object shadow by $\mathcal{V}_{\mathsf{obj}}$ (See Figure 4(a).) We define $\mathcal{V}^c_{\mathsf{obj}} = (\mathcal{V}_{\mathsf{obj}} \oplus \gamma_c) \setminus \mathcal{Q}$, as in Figure 4(b). Note that $\mathcal{V}_{\mathsf{obj}} \subset \mathcal{B} \cap \mathcal{V}^c_{\mathsf{obj}}$. The object shadow is the subset of $\mathcal{B} \setminus \mathcal{Q}$ that cannot be removed in directions d_p or $-d_p$. Thus, $\mathcal{V}_{\mathsf{obj}}$ should be removed in direction d_c and so we are interested in $\mathcal{V}^c_{\mathsf{obj}}$. Consider the illumination of $\mathbb{R}^3$ by light sources at infinity in directions d_p and $-d_p$ again. We treat both $\mathcal{Q}$ and $\mathcal{B} \cap \mathcal{V}^c_{\mathsf{obj}}$ as opaque and this yields a larger set of non-illuminated points. We call the subset of non-illuminated points in $\mathcal{B} \setminus (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}})$ the *extended shadow* (See Figure 4(c).) That is, for any point x in the extended shadow, the rays emitting from x towards d_p and $-d_p$ intersect $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$. We denote the extended shadow by $\mathcal{V}_{\mathsf{ext}}$. We define $\mathcal{V}^c_{\mathsf{ext}} = (\mathcal{V}_{\mathsf{ext}} \oplus \gamma_c) \setminus (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}})$. Note



(a) (b) (c)

Figure 4: Some definitions are illustrated in 2D: (a) the set $\mathcal{V}_{\mathsf{obj}}$, (b) the swept volume $\mathcal{V}^c_{\mathsf{obj}} = (\mathcal{V}_{\mathsf{obj}} \oplus \gamma_c) \setminus \mathcal{Q}$, and (c) the extended shadow $\mathcal{V}_{\mathsf{ext}}$ (the darker gray region.)

that $\mathcal{V}_{\mathsf{ext}} \subset \mathcal{B} \cap \mathcal{V}^c_{\mathsf{ext}}$. By definition, $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$ forbids the points in $\mathcal{V}_{\mathsf{ext}}$ to be removed in directions d_p or $-d_p$. So $\mathcal{V}_{\mathsf{ext}}$ should be removed in direction d_c. This explains why we are interested in $\mathcal{V}^c_{\mathsf{ext}}$.

Although the construction of $\mathcal{V}^c_{\mathsf{obj}}$ can be viewed as sweeping $\mathcal{V}_{\mathsf{obj}}$ in d_c, $\mathcal{V}^c_{\mathsf{obj}}$ needs not be monotone in direction d_c because $\mathcal{V}^c_{\mathsf{obj}}$ does not include $\mathcal{Q}$. Similarly, $\mathcal{V}^c_{\mathsf{ext}}$ needs not be monotone in d_c.

3.1 Technical lemmas

We need three technical lemmas in proving the characterization.

Lemma 1 *Assume that d_p is the upward vertical direction. Let x be a point in $\mathcal{V}^c_{\mathsf{ext}}$. There is a point y directly above or below x such that $y \in \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$ and $\mathrm{int}(xy) \subset \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$.*

Proof. Since $x \in \mathcal{V}^c_{\mathsf{ext}}$, there is a point $z \in \mathcal{V}_{\mathsf{ext}}$ such that $x \in z \oplus \gamma_c$. Since $z \in \mathcal{V}_{\mathsf{ext}}$, the two rays $z \oplus \gamma_p$ and $z \oplus -\gamma_p$ intersect $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$. One of these two rays must intersect $\mathcal{V}^c_{\mathsf{obj}}$; otherwise, both rays intersect $\mathcal{Q}$ and z would be in $\mathcal{V}_{\mathsf{obj}}$ instead of $\mathcal{V}_{\mathsf{ext}}$. Without loss of generality, let z' be the first point in $\mathcal{V}^c_{\mathsf{obj}}$ hit by $z \oplus \gamma_p$. It follows that $\mathrm{int}(zz') \subset \mathcal{V}_{\mathsf{ext}}$.

Because $z' \in \mathcal{V}^c_{\mathsf{obj}}$, $(z' \oplus \gamma_c) \setminus \mathcal{Q} \subset \mathcal{V}^c_{\mathsf{obj}}$. So there is a point y directly above x such that xy is a translated copy of zz' and $y \in \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$. Because $\mathrm{int}(zz') \subset \mathcal{V}_{\mathsf{ext}}$, $(\mathrm{int}(zz') \oplus \gamma_c) \setminus (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}) \subset \mathcal{V}^c_{\mathsf{ext}}$. It follows that $\mathrm{int}(xy) \subset \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. □

Lemma 1 is needed to prove the following result, which says that $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is monotone in direction d_p. Then, it follows from the result of Ahn et al. [4] that $\mathcal{B} \cap (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ is castable using a 2-part cast. We will show later that these two cast parts are also the cast parts for $\mathcal{Q}$.

Lemma 2 *$\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is monotone in direction d_p.*

Proof. Without loss of generality, assume that d_p is the vertical upward direction. Suppose that the lemma is false. Then there is a vertical line ℓ that intersects $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ in two consecutive disjoint intervals. Let s_1 denote the upper interval and let s_3 denote the lower interval. Let s_2 be the interval in $\ell \setminus (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ between s_1 and s_3.

It is impossible that both s_1 and s_3 contain points inside $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$. Otherwise, $s_2 \subset \mathcal{V}_{\mathsf{ext}}$, contradicting the fact that s_2 lies outside $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. So s_1 or s_3, say s_1, lies inside $\mathcal{V}^c_{\mathsf{ext}}$.

By Lemma 1, there is a point y directly above or below s_1 such that $y \in \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$ and $\mathrm{int}(L) \subset \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$, where L is the shortest line segment connecting s_1 and y. There are two cases to consider and we derive a contradiction in each case.

Case 1: y lies below s_1. Since $y \in \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$, y must lie below s_2. But then $s_2 \subseteq \mathrm{int}(L) \subset \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. This is a contradiction because s_2 lies outside $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$.

Case 2: y lies above s_1. Consider the possibility that s_3 contains a point z in $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$. In this case any upward (resp. downward) ray emitting from any point in s_2 must intersect $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$ at y (resp. z). But then $s_2 \subset \mathcal{V}_{\mathsf{ext}}$, contradicting the fact that s_2 lies outside $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. The remaining possibility is that s_3 lies inside $\mathcal{V}^c_{\mathsf{ext}}$. By Lemma 1, there is a point y' directly above or below s_3 such that $y' \in \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$ and $\mathrm{int}(L') \subset \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$, where L' is the shortest line segment containing s_3 and y'. If y' lies above s_3, then y' must lie above s_2 as well. It follows that $s_2 \subseteq \mathrm{int}(L') \subset \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$, a contradiction. If y' lies below s_3, then any upward (resp. downward) ray emitting from any point in s_2 must intersect $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$ at y (resp. y'). But then $s_2 \subset \mathcal{V}_{\mathsf{ext}}$, a contradiction again. ◻

Lemma 2 gives us a way to construct the two cast parts. It also implies that the side core should occupy the region $\mathcal{B} \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ so that the remaining cavity is exactly $\mathcal{Q}$. The following result shows that the side core indeed includes $\mathcal{B} \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$.

Lemma 3 *Given d_p and d_c, if $\mathcal{Q}$ is castable, then $\mathcal{B} \cap \mathcal{V}^c_{\mathsf{obj}} \subseteq \mathcal{C}_c$ and $\mathcal{B} \cap \mathcal{V}^c_{\mathsf{ext}} \subseteq \mathcal{C}_c$.*

Proof. For any point $x \in \mathcal{V}_{\mathsf{obj}}$, both $x \oplus \gamma_p$ and $x \oplus -\gamma_p$ intersect $\mathcal{Q}$, and so x cannot belong to $\mathcal{C}_r$ or $\mathcal{C}_b$. Thus, $\mathcal{V}_{\mathsf{obj}} \subseteq \mathcal{C}_c$. Since $\mathcal{Q}$ is castable, $\mathcal{C}_c \oplus \gamma_c$ does not intersect $\mathcal{Q}$, $\mathcal{C}_r$, or $\mathcal{C}_b$. This implies that $\mathcal{B} \cap \mathcal{V}^c_{\mathsf{obj}} \subseteq \mathcal{C}_c$.

Let x be a point in $\mathcal{V}_{\mathsf{ext}}$. Both $x \oplus \gamma_p$ and $x \oplus -\gamma_p$ intersect $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$ by definition. We have already shown that $\mathcal{V}_{\mathsf{obj}} \subseteq \mathcal{C}_c$. Thus, both $x \oplus \gamma_p$ and $x \oplus -\gamma_p$ intersect $\mathcal{Q} \cup (\mathcal{C}_c \oplus \gamma_c)$. If x belongs to $\mathcal{C}_r$ or $\mathcal{C}_b$, $\mathcal{C}_r \oplus \gamma_p$ or $\mathcal{C}_b \oplus -\gamma_p$ would intersect $\mathcal{Q} \cup (\mathcal{C}_c \oplus \gamma_c)$, contradicting the castability of $\mathcal{Q}$. So x belongs to $\mathcal{C}_c$, which implies that $\mathcal{V}_{\mathsf{ext}} \subseteq \mathcal{C}_c$. Again, because $\mathcal{Q}$ is castable, $\mathcal{C}_c \oplus \gamma_c$ does not intersect $\mathcal{Q}$, $\mathcal{C}_r$, or $\mathcal{C}_b$. Hence, $\mathcal{B} \cap \mathcal{V}^c_{\mathsf{ext}} \subseteq \mathcal{C}_c$. ◻

3.2 Characterization

We are ready to prove the necessary and sufficient condition for which an object is castable with respect to some given removal directions for the two cast parts and the side core.

Theorem 4 *Given d_p and d_c, $\mathcal{Q}$ is castable if and only if $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is monotone in d_c.*

Proof. Suppose that $\mathcal{Q}$ is castable. Assume to the contrary that $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is not monotone in d_c. Then, there is a line parallel to d_c that intersects $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ in two disjoint intervals. It follows that there exists a point x in $\mathcal{V}_{\mathsf{obj}} \cup \mathcal{V}_{\mathsf{ext}}$ such that $x \oplus \gamma_c$ intersects $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ in two disjoint intervals. If $x \in \mathcal{V}_{\mathsf{obj}}$, then $x \oplus \gamma_c \subset \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}}$; if $x \in \mathcal{V}_{\mathsf{ext}}$, then $x \oplus \gamma_c \subset \mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. So in either case $x \oplus \gamma_c$ must intersect $\mathcal{Q}$ in order that $x \oplus \gamma_c$ intersects $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ in two disjoint intervals. Since $\mathcal{Q}$ is castable, $\mathcal{C}_c \oplus \gamma_c$ avoids $\mathcal{Q}$. Because $x \in \mathcal{V}_{\mathsf{obj}} \cup \mathcal{V}_{\mathsf{ext}} \subset \mathcal{B} \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ which is a subset of $\mathcal{C}_c$ by Lemma 3, we conclude that $x \oplus \gamma_c$ avoids $\mathcal{Q}$, a contradiction. This proves the necessity of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ being monotone in d_c.

We prove the sufficiency by showing the construction of a cast for $\mathcal{Q}$. Ahn et al. [4] proved that an object is castable using two cast parts (without any side core) with removal directions d and $-d$ if and only if the object is monotone in direction d. Thus, Lemma 2 implies that $\mathcal{B} \cap (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ is castable using a 2-part cast with removal directions d_p and $-d_p$. We use the construction by Ahn et. al [4] to build $\mathcal{C}_r$ and $\mathcal{C}_b$ with the necessary modification for handling the core. Without loss of generality, assume that d_p is the upward vertical direction, d_c makes an angle of at most $\pi/2$ with d_p, and the horizontal projection of d_c aligns with the positive x-axis.

First, we make $\mathcal{B}$ sufficiently large and position $\mathcal{Q}$ inside $\mathcal{B}$ so that $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ intersects the interior of one vertical side facet of $\mathcal{B}$ only. Let S be that vertical side facet of $\mathcal{B}$. Thicken S slightly to form a slab S^+. Let T be the top horizontal facet of $\mathcal{B}$. Thicken T slightly to form one slab T^+. We can almost make $\mathcal{B} \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ the side core, but it may be disconnected. So we add S^+ to connect the components in $\mathcal{B} \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ to form the side core $\mathcal{C}_c$. Next, we follow the method in [4] to construct $\mathcal{C}_r \cup \mathcal{C}_b$ as a 2-part cast for $\mathcal{B} \cap (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$. Subtract $\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ from $(\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}) \oplus \gamma_p$. Intersect the resulting shape with $\mathcal{B}$ and denote the intersection by $\mathcal{X}$. We can almost make $\mathcal{X}$ the red cast part, but $\mathcal{X}$ may be disconnected. So we add $T^+ \setminus S^+$ to connect the components of $\mathcal{X}$ to form one red cast part $\mathcal{C}_r$. Lastly, we construct the blue cast part $\mathcal{C}_b$ as $\mathcal{B} \setminus (\mathcal{Q} \cup \mathcal{C}_r \cup \mathcal{C}_c)$.

We argue that $\mathcal{C}_r \oplus \gamma_p$, $\mathcal{C}_b \oplus -\gamma_p$, and $\mathcal{C}_c \oplus \gamma_c$ do not intersect $\mathcal{Q}$ or each other. Since $\mathcal{B} \cap (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ is castable using the 2-part cast $\mathcal{C}_r \cup \mathcal{C}_b$, $\mathcal{C}_r \oplus \gamma_p$ and $\mathcal{C}_b \oplus -\gamma_p$ do not intersect $\mathcal{Q}$ or each other. Since $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is monotone in d_c by assumption, $(\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}) \oplus \gamma_c$ does not intersect $\mathcal{Q}$. Also, $S^+ \oplus \gamma_c$ avoids $\mathcal{Q}$ by construction. So $\mathcal{C}_c \oplus \gamma_c$ does not intersect $\mathcal{Q}$. It remains to show that $\mathcal{C}_c \oplus \gamma_c$ avoids $\mathcal{C}_r \oplus \gamma_p$ and $\mathcal{C}_b \oplus -\gamma_p$.

Assume to the contrary that $\mathcal{C}_c \oplus \gamma_c$ intersects $\mathcal{C}_r \oplus \gamma_p$ or $\mathcal{C}_b \oplus -\gamma_p$ at a point z. By construction, $S^+ \oplus \gamma_c$ cannot intersect $\mathcal{C}_r \oplus \gamma_p$ or $\mathcal{C}_b \oplus -\gamma_p$. So $(\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}) \oplus \gamma_c$ intersects $\mathcal{C}_r \oplus \gamma_p$ or $\mathcal{C}_b \oplus -\gamma_p$ at z. Since $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is monotone in d_c by assumption, $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}} = (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}) \oplus \gamma_c$. So z lies in $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. By construction, S is the only (vertical) facet of $\mathcal{B}$ intersected by $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. If z lies outside $\mathcal{B}$, the support plane of S would separate z from $\mathcal{C}_r \oplus \gamma_p$ and $\mathcal{C}_b \oplus -\gamma_p$. Therefore, z lies in $\mathcal{B}$, which implies that $z \in \mathcal{B} \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}) \subset \mathcal{B} \cap (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$. However, this means that $\mathcal{C}_r \oplus \gamma_p$ or $\mathcal{C}_b \oplus -\gamma_p$ intersects $\mathcal{B} \cap (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$, contradicting the fact that $\mathcal{C}_r \cup \mathcal{C}_b$ is a 2-part cast for $\mathcal{B} \cap (\mathcal{Q} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$. □

If we are given a CAD system that is equipped with visibility computation, volume sweeping, and monotonicity checking operation, the characterization in Theorem 4 can be used directly to check the castability of any object. The proof also yields the construction of the cast.

4 An algorithm for polyhedra

We apply Theorem 4 to check the castability of a polyhedron. The goal is to obtain a discrete algorithm whose running time depends on the combinatorial complexity of the polyhedron. To be consistent with the previous section, our object is the interior of the polyhedron and we denote it by $\mathcal{P}$. The combinatorial complexity n of $\mathcal{P}$ is the number of vertices, edges, and facets in $\mathrm{bd}(\mathcal{P})$. We present an $O(n^3 \log n)$-time algorithm for testing the castability of $\mathcal{P}$ given d_p and d_c. During the verification, we compute $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$, from which the cast can be easily obtained as mentioned in the proof of Theorem 4.

Throughout this section, we assume that d_p is the upward vertical direction. We also make two assumptions about non-degeneracy. First, no facet in $\mathrm{bd}(\mathcal{P})$ is vertical. Second, the vertical projections of two polyhedron edges are either disjoint or they cross each other. These non-degeneracy assumptions simplify the presentation and they can be removed by a more detailed analysis. We call a facet of $\mathcal{P}$ an *up-facet* if its outward normal makes an acute angle with d_p, and a *down-facet*, otherwise.

Let $\mathcal{H}$ be a horizontal plane below $\mathcal{P}$. We project all facets of $\mathcal{P}$ onto $\mathcal{H}$. The projections may intersect each other and we insert vertices at the crossings. The resulting subdivision has $O(n^2)$ size and we denote it by $\mathcal{M}$. We associate with each cell of $\mathcal{M}$ the set of polyhedron facets whose projections cover it. We can compute $\mathcal{M}$ in $O(n^2 \log n)$ time using a plane-sweep algorithm. The association of polyhedron facets to cells can also be done in $O(n^3 \log n)$ time during the plane sweep. After computing $\mathcal{M}$, we test whether $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is monotone in d_c as follows (see Theorem 4).

We partition $\mathcal{H}$ into 2D slabs by taking vertical planes parallel to d_c through all vertices of $\mathcal{M}$. Since there are $O(n^2)$ vertices in $\mathcal{M}$ and a vertical plane parallel to d_c intersects $O(n)$ edges of $\mathcal{P}$, there are $O(n^3)$ intersections in total. So the overlay of $\mathcal{M}$ and the slabs can be computed in $O(n^3 \log n)$ time using a plane-sweep algorithm.

Consider a slab Σ on $\mathcal{H}$. By the construction, Σ contains no vertex in its interior and is partitioned into $O(n)$ regions by the edges of $\mathcal{M}$. Let d be the projection of d_c on $\mathcal{H}$. The regions in Σ are linearly ordered in direction d and we label them by Δ_0, Δ_1, ... in this order. Notice that Δ_0 is unbounded in direction $-d$ and the last region is unbounded in direction d. We use ζ_i to denote the boundary edge between Δ_{i-1} and Δ_i. For each region Δ_i, we keep the set of polyhedron facets whose projections cover it. We cannot do this straightforwardly. Otherwise, since there are $O(n^3)$ regions over all slabs and we may keep $O(n)$ polyhedron facets per region, the total time and space needed would be $O(n^4)$. The key observation is that if we walk from Δ_0 along Σ in direction d and record the changes in the set of facets whenever we cross a boundary edge ζ_i, then the total number of changes in Σ is $O(n)$. Therefore, we can use a persistent search tree [15] to store the sets of polyhedron facets for all regions in Σ. This takes $O(n \log n)$ time and $O(n)$ space to build per slab. Hence, it takes a total of $O(n^3 \log n)$ time and $O(n^3)$ space.

For each 2D slab Σ on $\mathcal{H}$, we employ an inductive strategy for testing the monotonicity of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ in d_c within the unbounded 3D slab $\Sigma \times [\infty, -\infty]$. Each region Δ_i in Σ gives rise to a trapezoidal pillar $\Delta_i \times [\infty, -\infty]$. We scan these trapezoidal pillars $\Delta_i \times [\infty, -\infty]$ in the order of $i = 0, 1, 2, \ldots$. We first discuss the data structures needed. Consider a boundary ζ_i. Take the vertical strip H_i through ζ_i. We translate H_i slightly into Δ_{i-1} (resp. Δ_i) and denote the perturbed strip by H_i^- (resp. H_i^+). Let I_i^- denote the intersection $H_i^- \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$ and let I_i^+ denote the intersection $H_i^+ \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$. Both I_i^- and I_i^+ consist of $O(n)$ trapezoids. Let τ be a trapezoid in I_i^- or I_i^+. We call the upper and lower sides of τ its *ceiling* and *floor*, respectively. The ceiling of τ lies on a boundary facet of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. We call this boundary facet the *ceiling-facet* of τ. This ceiling-facet may lie within a down-facet in $\mathrm{bd}(\mathcal{P})$ or it may be parallel to d_c and not contained in $\mathrm{bd}(\mathcal{P})$. (The latter kind of boundary facets are generated by the Minkowski sum of $\mathcal{V}_{\mathsf{obj}}$ or $\mathcal{V}_{\mathsf{ext}}$ with γ_c.) Therefore, it suffices to store a polyhedron facet or a plane parallel to d_c to represent the ceiling-facet. We denote this representation by $f_u(\tau)$. Similarly, the floor of τ lies on a boundary facet of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. This boundary facet may lie within an up-facet in $\mathrm{bd}(\mathcal{P})$ or it may be parallel to d_c and not contained in $\mathrm{bd}(\mathcal{P})$. We call it the *floor-facet* of τ and denote its representation by $f_\ell(\tau)$.

We incrementally grow a volume $\mathcal{V}^c$ during the scanning. The volume $\mathcal{V}^c$ is initially empty. If $\mathcal{P}$ is castable, $\mathcal{V}^c$ will be equal to $(\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}) \cap (\Sigma \times [\infty, -\infty])$ in the end. Consider the event that we cross the boundary ζ_i during the scanning. Suppose that the portion of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ within $\bigcup_{j=0}^{i-1} \Delta_j \times [\infty, -\infty]$ is monotone in d_c. We first check whether the portion of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ inside $\Delta_i \times [\infty, -\infty]$ will change the monotonicity. Later, we discuss how to grow $\mathcal{V}^c$ if the monotonicity is preserved. Note that there is a difference in the sets of polyhedron facets whose projections cover Δ_{i-1} and Δ_i. There are several cases.

1. For each trapezoid $\tau \in I_i^-$, neither $f_u(\tau)$ nor $f_\ell(\tau)$ is about to vanish above ζ_i. Let e be the polyhedron edge e that projects vertically onto ζ_i.

   (a) I_i^- is empty. Let f and f' be the two polyhedron facets incident to e. If the vertical projection of f or f' covers Δ_{i-1}, there is nothing to be done. Otherwise, the vertical projections of f and f' cover Δ_i but not Δ_{i-1}. Assume that f is higher than f' locally around e. If f is an up-facet, there is nothing to be done. Suppose that f is a down-facet. So f and f' bound a subset of $\mathcal{V}_{\mathsf{obj}}$ locally around e above Δ_i. If the outward normal of f or f' makes an obtuse angle with d_c, we abort and report that $\mathcal{P}$ is not castable. The reason is that f or f' must intersect the Minkowski sum of this subset of $\mathcal{V}_{\mathsf{obj}}$ around e with γ_c. So $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is not monotone in d_c and $\mathcal{P}$ is not castable by Theorem 4. If the outward normals of f and f' make non-obtuse angles with d_c, we insert a new trapezoid τ into I_i^+. We set $f_u(\tau) = f$ and $f_\ell(\tau) = f'$.

   (b) I_i^- is non-empty. If the vertical projection of a polyhedron facet incident to e covers Δ_{i-1}, there is nothing to be done. Suppose that the vertical projections of the two polyhedron facets incident to e cover Δ_i but not Δ_{i-1}. Consider the projection e^- of e in direction $-d_c$ onto H_i^-. Since $\mathcal{P} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is monotone in d_p by Lemma 2, the space between two trapezoids in I_i^- is the polyhedron interior. Thus the projection e^- cannot lie between two trapezoids in I_i^-. So there are only two cases:

      i. The projection e^- cuts across the interior of a trapezoid $\tau \in I_i^-$. In this case, we abort and report that $\mathcal{P}$ is not castable. The reason is that a polyhedron facet incident to e must intersect $\tau \oplus \gamma_c$, and so $(\tau \oplus \gamma_c) \setminus \mathcal{P}$ is not monotone in d_c. It follows that $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is not monotone in d_c and so $\mathcal{P}$ is not castable by Theorem 4.

      ii. The projection e^- lies above all trapezoids in I_i^-. The case that e^- lies below all trapezoids can be handled symmetrically. Let f be the down-facet incident to e. If we project e vertically downward, the projection either lies on some up-facet f', or a boundary facet of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ that is parallel to d_c. Let e' denote this vertical downward projection of e.

         A. If e' lies on an up-facet f', e and e' define a new trapezoid τ that lies above all trapezoids in I_i^-. We set $f_u(\tau) = f$ and $f_\ell(\tau) = f'$. I_i^+ contains all trapezoids in I_i^- as well as τ.

         B. If e' lies on a boundary facet of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ that is parallel to d_c, then e' actually lies on $f_u(\tau)$ where τ is the topmost trapezoid in I_i^-. Thus, we should grow τ upward and set $f_u(\tau) = f$. I_i^+ contains this updated trapezoid τ and the other trapezoids in I_i^-.

After the above update, we check the outward normal of f. If this normal makes an obtuse angle with d_c, then f intersects $\tau \oplus \gamma_c$ and we should abort and conclude as before that $\mathcal{P}$ is not castable.

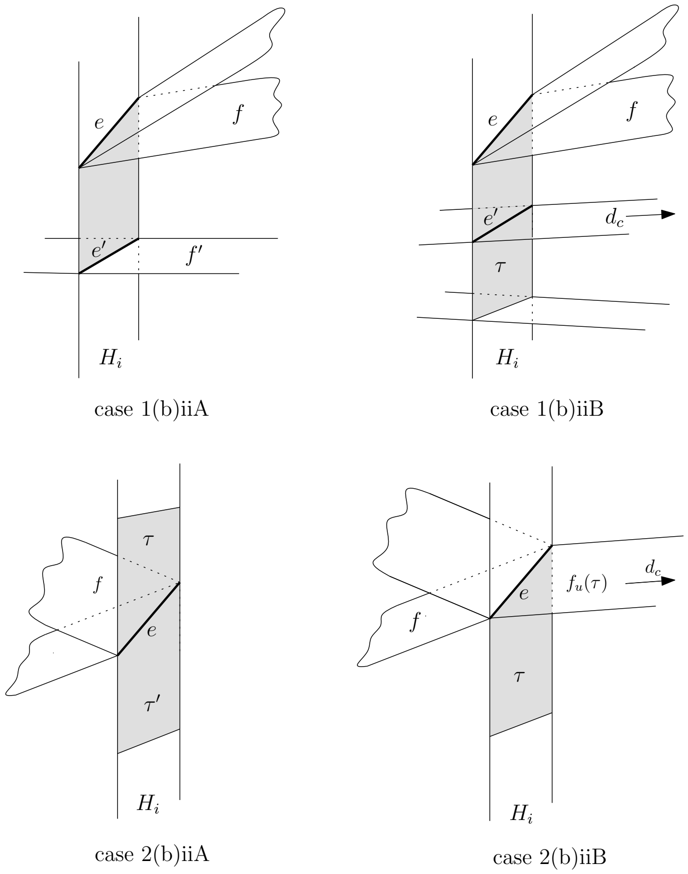


Figure 5: Two cases of 1(b)ii and two cases of 2(b)ii. The gray trapezoids are subsets of $H_i \cap (\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}})$.

2. For some trapezoid $\tau \in I_i^-$, $f_u(\tau)$ or $f_\ell(\tau)$ is about to vanish above ζ_i. Notice that if the ceiling or floor facet of τ is a boundary facet of $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$, it cannot be intercepted by the interior of a polyhedron facet in direction d_c. Otherwise, $\mathcal{P} \cup \mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ would not be monotone in d_p, contradicting Lemma 2. Therefore, the vanishing $f_u(\tau)$ or $f_\ell(\tau)$ must end at the polyhedron edge e that projects vertically onto ζ_i. There are two cases:
   (a) The vanishing $f_u(\tau)$ or $f_\ell(\tau)$ is not a polyhedron facet incident to e. In this case the vanishing $f_u(\tau)$ or $f_\ell(\tau)$ is intercepted by the interior of $\mathcal{P}$ at e. This means that $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is not monotone in d_c and so $\mathcal{P}$ is not castable as discussed before.
   (b) The vanishing $f_u(\tau)$ or $f_\ell(\tau)$ is a polyhedron facet incident to e. There are two cases:
      i. The polyhedron facets incident to e lie locally on different sides of the vertical plane through ζ_i. Let f be the facet incident to e that lies locally in direction d_c from e.

In this case, the vanishing $f_u(\tau)$ or $f_\ell(\tau)$ should be replaced by f. However, if the outward normal of f makes an obtuse angle with d_c, then f intersects $\tau \oplus \gamma_c$ and we should abort as $\mathcal{P}$ is not castable.

ii. Otherwise, both incident facets of e lie locally in direction $-d_c$ from e. There is no change in monotonicity status, but we need to perform update as follows. Let f be the vanishing $f_u(\tau)$ or $f_\ell(\tau)$ of τ. There are two cases:

A. There are trapezoids in I_i^- that lie above and below f. Clearly, τ is one of them. Let τ' be the other trapezoid. Then $f_u(\tau')$ or $f_\ell(\tau')$ is about to vanish above ζ_i too. In this case, we should merge τ and τ' into one trapezoid. The ceiling-facet and floor-facet of this merged trapezoid are the non-vanishing ceiling-facet and floor-facet of τ and τ'. I_i^+ contains this merged trapezoid and the trapezoids in I_i^- other than τ and τ'.

B. All trapezoids in I_i^- lie on one side of f. Assume that τ is the topmost trapezoid in I_i^-. The other case can be handled symmetrically. Then $f = f_u(\tau)$. It means that we are about to sweep the shadow volume below f and bounded by τ into the space above Δ_i. Thus, we should set $f_u(\tau)$ to be the plane that passes through e and is parallel to d_c. I_i^+ contains this updated trapezoid τ and the other trapezoids in I_i^-.

By representing each trapezoid in I_i^- combinatorially by its ceiling-facet and floor-facet, the above description tells us how to update I_i^- combinatorially to produce I_i^+. Notice that I_i^+ will be treated as I_{i+1}^- when we are about to cross the boundary ζ_{i+1} in the future. By storing the trapezoids in I_i^- in a balanced binary search tree, the update at ζ_i can be performed in $O(\log n)$ time. Since there are $O(n)$ regions in Σ, scanning Σ takes $O(n \log n)$ time. Summing over all 2D slabs on $\mathcal{H}$ gives a total running time of $O(n^3 \log n)$.

What about growing $\mathcal{V}^c$ into $\Delta_i \times [\infty, -\infty]$? After the update, for each trapezoid $\tau \in I_i^+$, $f_u(\tau)$ and $f_\ell(\tau)$ cut $\Delta_i \times [\infty, -\infty]$ into two unbounded solids and one bounded solid B_τ. Conceptually, we can grow $\mathcal{V}^c$ by attaching B_τ for each trapezoid $\tau \in I_i^+$, but this is too consuming. Observe that if I_i^+ merely inherits a trapezoid τ from I_i^-, there is no hurry to attach B_τ right now. Instead, we wait until ζ_j for the smallest $j > i$ such that I_j^+ does not inherit τ from I_{j-1}^-. Then $f_u(\tau)$ and $f_\ell(\tau)$ cut $R \times [\infty, -\infty]$ into two unbounded solids and one bounded solid S_τ, where R is the area within Σ bounded by ζ_i and ζ_j. We attach S_τ to grow $\mathcal{V}^c$. By adopting this strategy, we spend $O(1)$ time to grow $\mathcal{V}^c$ when we cross a region boundary. Hence, we spend a total of $O(n^3)$ time to construct $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$. Once $\mathcal{V}^c_{\mathsf{obj}} \cup \mathcal{V}^c_{\mathsf{ext}}$ is available, we can construct the cast in $O(n^3)$ time as explained in the proof of Theorem 4.

Theorem 5 *Given d_p and d_c, the castability of a polyhedron with size n can be determined in $O(n^3 \log n)$ time and $O(n^3)$ space. If castable, the cast can be constructed in the same time and space bounds.*

5 Experimental results

We developed a preliminary implementation of the algorithm of Theorem 5. Figure 6 shows the output of our implementation on some polyhedra: the direction d_p is the upward vertical direction

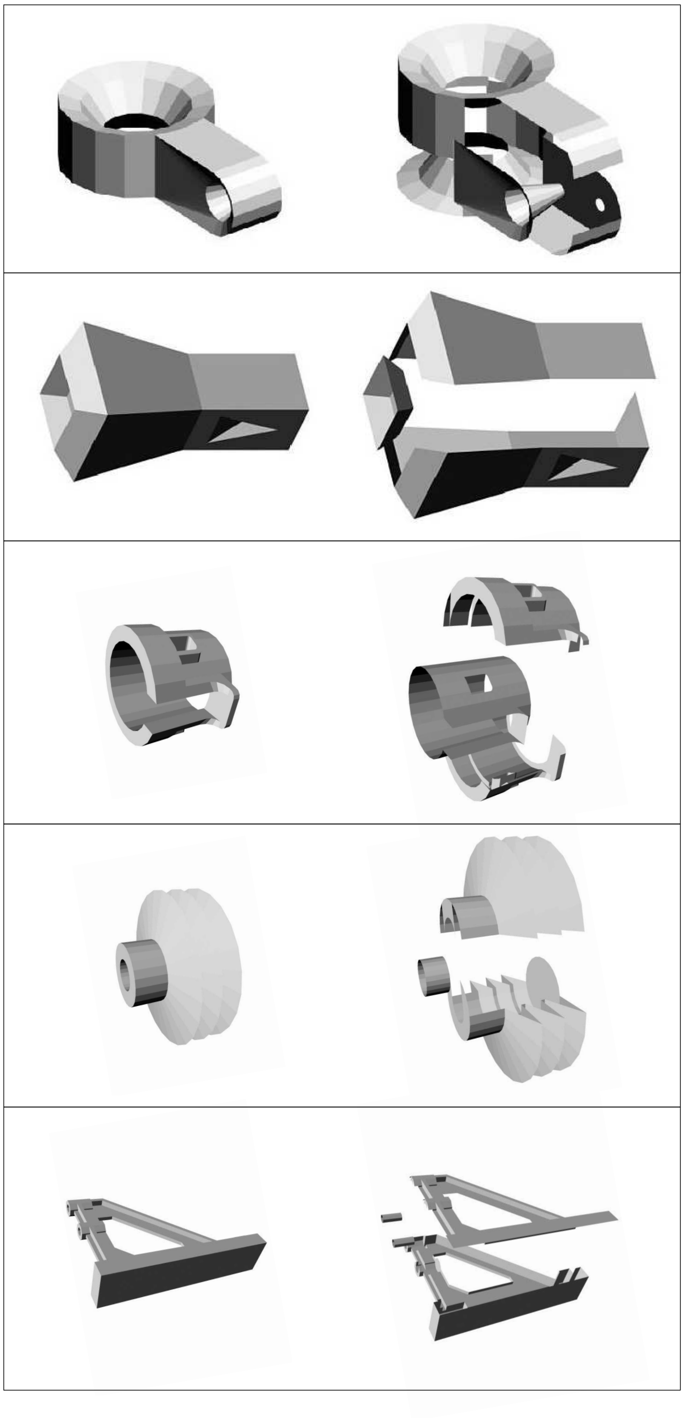


Figure 6: Objects and their boundary partitions into three groups by the implementation of our algorithm. The first two are artificial objects. The next three are a part of a camera body, a screw, and a wall bracket in order.

and the direction d_c is the leftward direction. In the figure, the boundary of each object is partitioned into three groups depending on which cast part they belong to. For the ease of visualization, each boundary group is translated slightly in its corresponding removal direction.

6 Worst-case example

We present a lower bound construction showing that a castable polyhedron of size n can require a cast of $\Omega(n^3)$ size. Thus the space complexity in Theorem 5 is worst-case optimal and the time complexity of our algorithm is at most a $\log n$ factor off the worst-case optimum. Throughout this section, we assume that d_p is the upward vertical direction and d_c is the leftward direction.

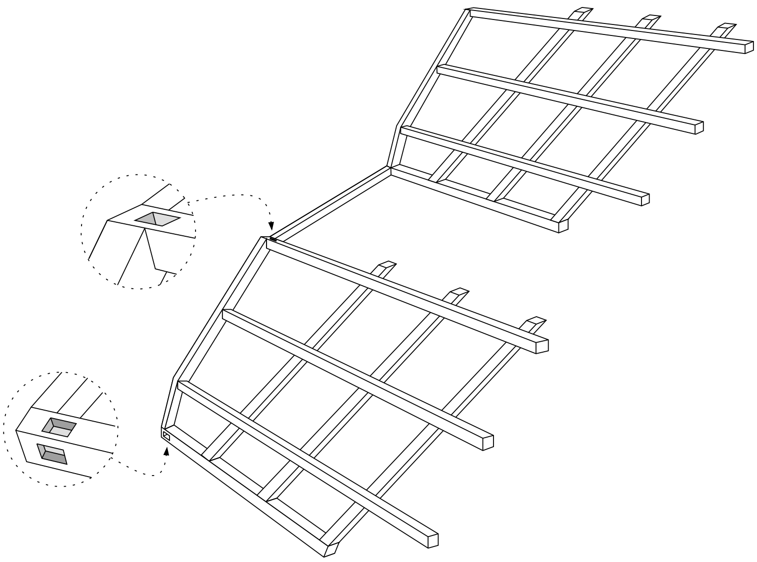
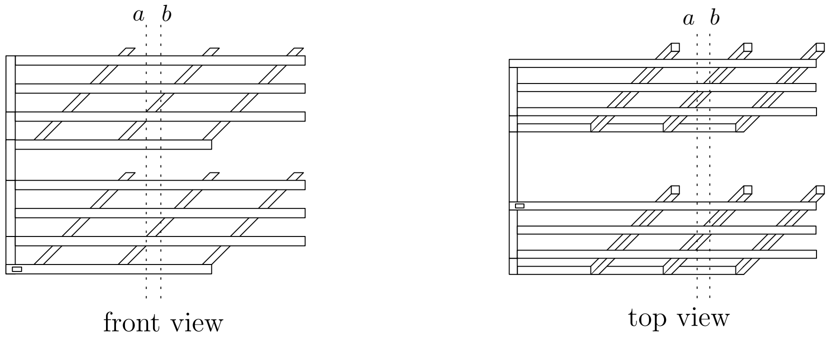


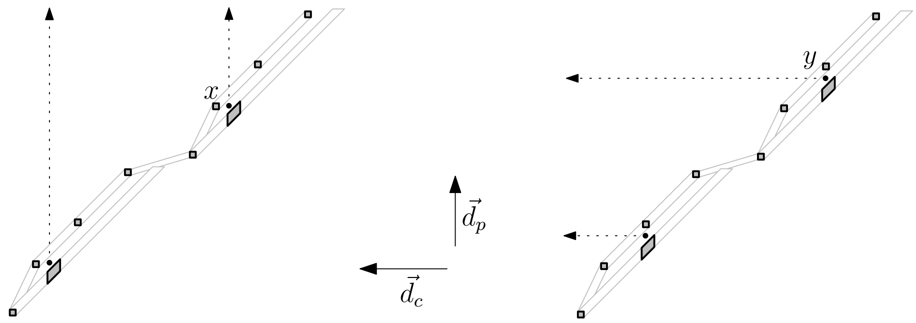
Figure 7: The lower bound example in a perspective view.

Figure 7 shows our lower bound construction. The polyhedron consists of two parts: the upper part has four horizontal legs in a staircase and three slanted legs sitting on a horizontal leg. The lower part is an almost identical copy of the upper part, except that it has three small holes as shown in the figure. The upper hole can only be covered by the red cast part to be removed vertically upward, and the other two holes can only be covered each by the side core and the blue cast part. Figure 8(a) shows the front view (when we look at Figure 7 from the left) and the top view of the polyhedron $\mathcal{P}$. In both projections, all three horizontal legs cross the other three slanted legs in the upper part as well as in the lower part.

Clearly, the polyhedron is castable with respect to the given directions d_p and d_c. We argue that the cast has $\Omega(n^3)$ size. Imagine that we take a plane h parallel to d_p and d_c, and sweep $\mathcal{P}$ with h in the direction orthogonal to h. During the sweep, the cross-section $\mathcal{P} \cap h$ is 2-part castable (in 2D) with parting direction d_p, except when h intersects a volume in $\mathcal{V}_{\mathsf{obj}}$. When this happens, $\mathcal{P} \cap h$ is castable in direction d_p if we add a side core to be removed in direction d_c. Figure 8(b) shows two cross-sections during the sweep. The left and right pictures in Figure 8(b) show the cross-section at positions a and b, respectively, shown in Figure 8(a). In the left cross section, let x



(a)



(b)

Figure 8: (a) A top view and a side view of the lower bound construction. (b) Two cross sections along a (left) and b (right). The only way to remove x (resp. y) is translating it in d_p (resp. d_c).

be a point in $\mathcal{B} \setminus \mathcal{P}$, lying in between a horizontal leg and a slanted leg in direction d_c and above the slanted leg. The only way to remove x is translating it in d_p, which means that the set of points in $\mathcal{B} \setminus \mathcal{P}$ hit by the ray from x in direction d_p to infinity must belong to $\mathcal{C}_r$. Now let y be a point in $\mathcal{V}_{\mathsf{obj}}$ in the right cross section. Analogously, the set of points in $\mathcal{B} \setminus \mathcal{P}$ hit by the ray from y in direction d_c to infinity must belong to $\mathcal{V}^c_{\mathsf{obj}}$ and hence $\mathcal{C}_c$. In addition, the set of points in $\mathcal{B} \setminus \mathcal{P}$ that lies in between the ray and $\mathcal{P}$ in direction d_p belong to $\mathcal{C}_c$ since these points belong to the set $\mathcal{V}_{\mathsf{ext}}$ by definition. While we sweep $\mathcal{P}$ with h, these two kinds of cross sections appear alternately.

We put $\Theta(n)$ horizontal legs and $\Theta(n)$ slanted legs in both the upper and the lower parts. In the upper part, each slanted leg must be in contact with both $\mathcal{C}_r$ and $\mathcal{C}_c$. Moreover, the contacts with $\mathcal{C}_r$ and $\mathcal{C}_c$ alternate $\Theta(n)$ times along the slanted leg. As a result, the slanted legs in the upper part have a total of $\Theta(n^2)$ contacts with $\mathcal{C}_c$. These contacts sweep in direction d_c and generate $\Theta(n^2)$ swept volumes. All these swept volumes belong to $\mathcal{C}_c$. The merging of any two such swept volumes is forbidden by the alternate appearances of the left cross-section in Figure 8(b); otherwise, some upward vertical ray (which lies inside $\mathcal{C}_r$) would block the removal of $\mathcal{C}_c$. Each swept volume projects vertically and produces a shadow on each horizontal leg that lies below it. Thus, the total complexity of $\mathcal{C}_c$ is $\Omega(n^3)$.

7 Conclusion

Given the removal directions for the two cast parts and the side core, we presented a characterization of the castability of an object, assuming that the swept volumes generated by the cast parts and the side core are required not to intersect the object or each other. Based on this characterization, we developed an $O(n^3 \log n)$-time and $O(n^3)$-space algorithm for testing a polyhedron of size n. We presented a lower bound construction to show that our cast complexity is worst-case optimal. Further research is needed to handle more than one core and to allow the core(s) to be removed first. Other interesting research problems include minimizing the complexity of the cast, finding all feasible removal directions for the cast parts and the side core, as well as allowing the cast parts to be removed in non-opposite directions.

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