Constructing Optimal Axis-Parallel Highways

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November 3, 2005

Abstract

For two points p and q in the plane, a line h—the highway—and a real $v > 1$, we define the *travel time* (also known as the *City distance*) from p and q to be the time needed to traverse a quickest path from p to q, where distances are measured in the underlying metric, and the speed on h is v and elsewhere 1. Given a set S of n points in the plane and a highway speed v, we consider the problem of finding an axis-parallel line that minimizes the maximum travel time over all pairs of points in S. In the case of the L_1-metric, we achieve a linear time algorithm. In the case of the Euclidean metric, our algorithms run in time $O(n \log n)$. We also show that placing k parallel highways does not reduce the maximum travel time.

1 Introduction

We give an algorithm that computes an optimal vertical highway. Clearly the same algorithm can also be used to find an optimal horizontal highway. We decided to describe the vertical case since it allows us to embed the points and their travel-time graphs (defined below) into the same plane.

Let $d(p, q)$ be the distance between p and q in the underlying metric. Throughout the paper we assume that S contains at least three points and that not all points have the same y-coordinate.

2 The Optimal vertical highway under the L_1 metric

For two points $p = (p_x, p_y)$ and $q = (q_x, q_y)$ in the plane and a vertical highway $h(x)$ at x-coordinate x with speed $v > 1$, we define the *highway distance* $\mathrm{hw}_x(p, q)$ of p and q to be the time needed to traverse a quickest path from p via $h(x)$ to q. Note that

$$\mathrm{hw}_x(p, q) = |p_x - x| + |q_x - x| + |p_y - q_y|/v.$$

Then the travel time $t_{pq}(x)$ from p to q in the presence of highway $h(x)$ is the minimum of the L_1-distance $d(p, q)$ and the highway distance $\mathrm{hw}_x(p, q)$. Note that the graph of the function $t_{pq}(x)$ that maps the x-coordinate of a vertical highway $h(x)$ to the travel time from p to q is a piecewise linear and continuous function consisting of at most 5 segments: let $v' = (v-1)/(2v)$, $\Delta y = |q_y - p_y|$, and assume $p_x \le q_x$, then

$$\ell_{pq} = p_x - \Delta y \cdot v' \quad \text{and} \quad r_{pq} = q_x + \Delta y \cdot v'$$

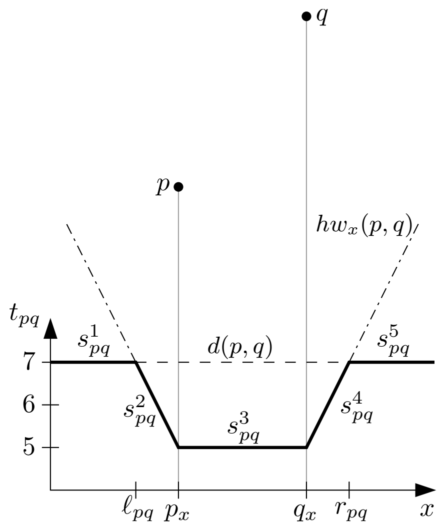

Figure 1: The travel time t_{pq} as a function of the x-coordinate of a highway with speed $v = 2$ such that $t_{pq}(x) = \min\{d(p,q), \mathrm{hw}_x(p,q)\}$. The graph of $d(p,q)$ is dashed, and the graph of $\mathrm{hw}_x(p,q)$ is dash dotted.

are the two x-coordinates where $d(p,q) = \mathrm{hw}_x(p,q)$. The travel-time graph of t_{pq} (see Figure 1) consists of three horizontal segments

$$s^1_{pq} = (-\infty : \ell_{pq}]\times\{d(p,q)\}, \quad s^3_{pq} = [p_x : q_x]\times\{\mathrm{hw}_x(p,q)\}, \quad s^5_{pq} = [r_{pq} : \infty)\times\{d(p,q)\}$$

and two line segments s^2_{pq} and s^4_{pq} that connect the segment s^1_{pq} to s^3_{pq} and the segment s^3_{pq} to s^5_{pq}, respectively. The slopes 2 and -2 of the non-horizontal segments s^2_{pq} and s^4_{pq}, respectively, do not depend on the highway speed v. We refer to s^3_{pq} as the *valley floor* and to the two other horizontal segments s^1_{pq} and s^5_{pq} as *plateaus* of the graph of t_{pq} or simply of $\{p,q\}$.

Our main problem in this paper is: given a set S of n points in the plane, our goal is to find a vertical highway that minimizes the maximum travel time over all pairs of points in S. The distance function $\omega(x) : \mathbb{R} \to \mathbb{R}$ of S is defined as

$$\omega(x) := \max_{\{p,q\}\subseteq S} t_{pq}(x).$$

Our problem is thus to find a placement $x \in \mathbb{R}$ that minimizes $\omega(x)$. We call such a placement a *goal placement.*

We first look at the graphs of t_{pq} of all pairs $\{p,q\} \subseteq S$. Their upper envelope of the graphs is represented by the function $\omega(x)$ that maps x to the maximum travel time over all pairs $\{p,q\}$, see Figure 2. Thus a global minimum of the upper envelope corresponds to a highway position that minimizes the maximum travel time.

A nonnegative function f is called *unimodal* if for any $c > 0$ the set $\{x \mid \mathcal{E}(x) \geqslant c\}$ is an interval. In other words, there exists $a \in \mathbb{R}$ such that f is increasing on $(-\infty : a]$ and decreasing on $[a : \infty)$. Our algorithm is based on the unimodality of the monovariate distance function.

Lemma 1 *The function $-\omega(x)$ is unimodal.*

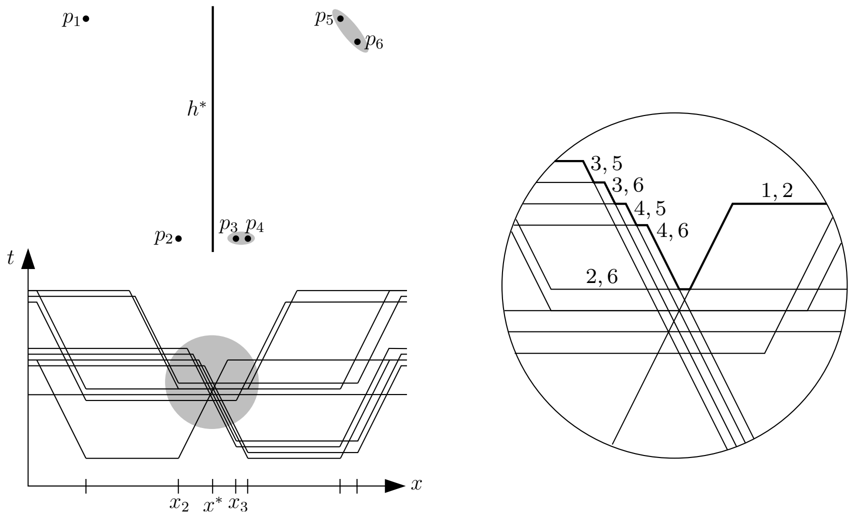

Figure 2: Time travel graphs of 6 points. The complexity of the upper envelope can be quadratic by adding $\lceil \frac{n-6}{2} \rceil$ points along the segment p_3p_4 and $\lfloor \frac{n-6}{2} \rfloor$ points along the segment p_5p_6.

Proof. All functions $-t_{pq}(x)$ are unimodal, thus their point-wise minimum, $-\omega(x)$, is unimodal, too. This is due to the fact that the set $\{x \mid \omega(x) \leq c\}$ is the intersection of the sets $\{x \mid t_{pq}(x) \leq c\}$ over all pairs $\{p, q\}$, and the intersection of intervals is an interval. $\square$

Let x^* be the leftmost global minimum. It is clear that x^* is bounded if not all points lie on the same horizontal line. Let h^* be the highway with x-coordinate x^* and let $t^* = \max_{\{p,q\} \subseteq S} t_{pq}(x^*)$ be the maximum travel time given h^*. Our goal is to compute x^* and t^* efficiently.

The complexity of the upper envelope can be superlinear in the number of graphs: there are $O(n^2)$ graphs, each of which consists of at most 5 line segments. The maximum complexity of the upper envelope of n^2 segments can be $O(n^2\alpha(n))$ [4], where $\alpha(n)$ is the inverse function to the Ackermann function. There are indeed time travel graphs of n points whose upper envelope has quadratic complexity: Figure 2 shows time travel graphs of 6 points. Imagine that we add $\lceil \frac{n-6}{2} \rceil$ points along the segment p_3p_4 and $\lfloor \frac{n-6}{2} \rfloor$ points along the segment p_5p_6, then the upper envelope has quadratic complexity. Note that the part of the upper envelope in the interval $[x_2 : x_3]$ of two consecutive x-coordinates of input points has quadratic complexity. Even when we locate the interval of x-coordinates of input points that contains the global optimum, we may still need to search the optimum over the upper envelope with quadratic complexity.

2.1 Highway distance only

We now turn our attention to the special case, where we force each pair to travel via the highway, that is, $t_{pq}(x) = \mathrm{hw}_x(p, q)$ for $x \in \mathbb{R}$. We consider all the graphs of highway distances $\mathrm{hw}_x(p, q)$ for all pairs $\{p, q\}$ of points in S, and let $\mathcal{E}$ denote their upper envelope. Recall that we view $\mathcal{E}$ as the function $\omega(x) = \max_{\{p,q\} \subseteq S} \mathrm{hw}_x(p, q)$

that maps $\mathbb{R}$ to $\mathbb{R}$. Note that the function $-\omega(x)$ is unimodal.

Lemma 2 *Let $\{a,b\}$ and $\{c,d\}$ be two pairs of points, and $h(x_0)$ be a highway such that $\mathrm{hw}_{x_0}(a,b) = \mathrm{hw}_{x_0}(c,d)$ with $\max\{a_x, b_x\} < x_0 < \min\{c_x, d_x\}$. There always exists a pair $\{\alpha,\beta\}$ such that $\alpha \in \{a,b\}$ and $\beta \in \{c,d\}$, and $\mathrm{hw}_{x_0}(\alpha,\beta) \geqslant \mathrm{hw}_{x_0}(a,b)$.*

Proof. Without loss of generality we assume $a_y > b_y$ and $c_y > d_y$. Let Π_{ab} and Π_{cd} be two shortest paths that connect a to b, and c to d via the highway $h(x_0)$, respectively. Let $I = \Pi_{ab} \cap \Pi_{cd}$. There are two cases, either $I \neq \emptyset$ or $I = \emptyset$, see Figure 3 and Figure 4.

For $r \in \{a,b,c,d\}$, let r^h be the point on the highway $h(x_0)$ with with y-coordinate r_y. For each of these cases, we define three values

$$\sigma_1 = (d_y^h - b_y^h)/v, \qquad \sigma_2 = (b_y^h - c_y^h)/v, \qquad \sigma_3 = (c_y^h - a_y^h)/v.$$

Then, for both cases, $\mathrm{hw}_{x_0}(b,c) = |bb^h| + |cc^h| - \sigma_2$ and $\mathrm{hw}_{x_0}(c,d) = |cc^h| + |dd^h| - \sigma_1 - \sigma_2$. If $\mathrm{hw}_{x_0}(b,c) \geqslant \mathrm{hw}_{x_0}(c,d)$, then we are done. Therefore, we assume that $\mathrm{hw}_{x_0}(b,c) < \mathrm{hw}_{x_0}(c,d)$. From this, we get $|bb^h| < |dd^h| - \sigma_1$.

We are going to show that $\mathrm{hw}_{x_0}(a,b) < \mathrm{hw}_{x_0}(a,d)$. Note that $\mathrm{hw}_{x_0}(a,d) = |aa^h| + |dd^h| - \sigma_1 - \sigma_2 - \sigma_3$ if $I \neq \emptyset$, and $\mathrm{hw}_{x_0}(a,d) = |aa^h| + |dd^h| + \sigma_1 + \sigma_2 + \sigma_3$ if $I = \emptyset$.

$$\begin{aligned} \mathrm{hw}_{x_0}(a,b) &= |aa^h| + |bb^h| - \sigma_2 - \sigma_3 \\ &< |aa^h| + |dd^h| - \sigma_1 - \sigma_2 - \sigma_3 \\ &\leqslant \mathrm{hw}_{x_0}(a,d). \end{aligned}$$

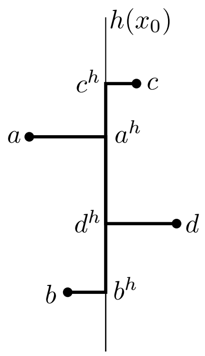

Figure 3: A tie with $\Pi_{ab} \cap \Pi_{cd} \neq \emptyset$.

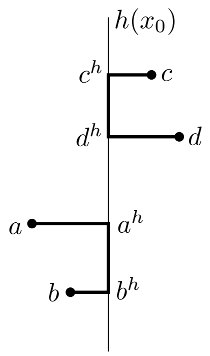

Figure 4: A tie with $\Pi_{ab} \cap \Pi_{cd} = \emptyset$.

□

The previous lemma, together with the unimodality of $-\omega(x)$, immediately implies the following corollary.

Corollary 1 *The x-interval of the minimum of $\mathcal{E}$ lies in the maximal x-interval of the highest valley floor.*

Proof. By Lemma 1, there exists a single unique interval of the minimum of $\mathcal{E}$. Assume to the contrary that this interval of highway positions does not lie in the maximal interval $[p_x : q_x]$ of the valley floor with maximum highway distance $\mathrm{hw}_{x\in[p_x:q_x]}(p,q)$. By Lemma 2 it must lie in the interval of some valley floor with a travel distance at most $\mathrm{hw}_{x\in[p_x:q_x]}(p,q)$. But the travel-time graph $t_{pq}(x)$ has travel time strictly larger than $\mathrm{hw}_{x\in[p_x:q_x]}(p,q)$ for any $x \notin [p_x : q_x]$, and the valley floor does not appear on the upper envelope of all travel-time graphs, which contradicts Lemma 1. □

2.2 Algorithms

Recall that $\mathrm{hw}_x(p,q) = |p_x - x| + |q_x - x| + |p_y - q_y|/v$ whose time graph consists of a valley floor s^3_{pq}, a *left ray* $(-\infty : p_x] \times \{\mathrm{hw}_x(p,q)\}$, and a *right ray* $[q_x, \infty) \times \{\mathrm{hw}_x(p,q)\}$.

Lemma 3 *The x-interval of the minimum of $\mathcal{E}$ realizes the optimal highway placement.*

Proof. Clearly, the highest valley floor of all travel-time graphs is still the highest valley floor in the all highway time graphs.

Assume to the contrary that the highest valley floor doesn't appear on the upper envelope of all highway time graphs. Since the graph of $\mathrm{hw}_x(p,q)$ coincides with the graph of $t_{pq}(x)$, except in two intervals $(-\infty : \ell_{pq})$ and (r_{pq}, ∞), where $\mathrm{hw}_x(p,q) > t_{pq}(x)$, this happens only when, in the interval $[p_x : q_x]$, (a) a left (right) ray lies above s^3_{pq} in the interval $[p_x : q_x]$, or (b) the upper envelope of a left ray and a right ray lies above s^3_{pq}. For the case (a), the left (right) ray must cross the right (left) ray of $\mathrm{hw}_x(p,q)$ at a point strictly higher than s^3_{pq}. Lemma 2 shows that there exists another valley floor with at least the time of the crossing. For the case (b), again Lemma 2 shows that there exists another valley floor with at least the time of the crossing. Therefore we have contradiction that s^3_{pq} is not highest. □

The algorithm consists of two steps: first we compute the pair $\{p,q\}$ of points that defines the maximal interval of the highest valley floor. From this we get the x-interval $[p_x : q_x]$ and its highway distance at the valley floor. In the second step, we compute the rightmost left ray of highway graphs, including the left ray of the graph of $\mathrm{hw}_x(p,q)$. This left ray intersects the highest valley floor at a point on $\mathcal{E}$, which is an optimal highway placement.

The diametral pair under the L_1 metric is one of the two pairs of opposite points extreme in the directions of vectors $(\pm 1, \pm 1)$. Imagine now that we have a vertical highway with speed $v > 1$ between every pair of points. Then the diametral pair can still be found in linear time as follows: we compute 4 points p_1, p_2, p_3 and p_4 extreme in the directions of vectors $(\pm 1, \pm v)$. These points can be found by linear search. Then one of the two pairs of opposite points is the diametral pair. This diametral pair has the maximum highway distance among all pairs separated by a highway. Therefore, it defines the highest valley floor in the travel-time graphs of all pairs.

Without loss of generality, assume now that all points have positive x-coordinates, and the vertical highway lies at $x = 0$ (See Figure 5.) We define two functions f and g as follows: for a point p, $f(p) := p_x + p_y/v$ and $g(p) := p_x - p_y/v$. Then for

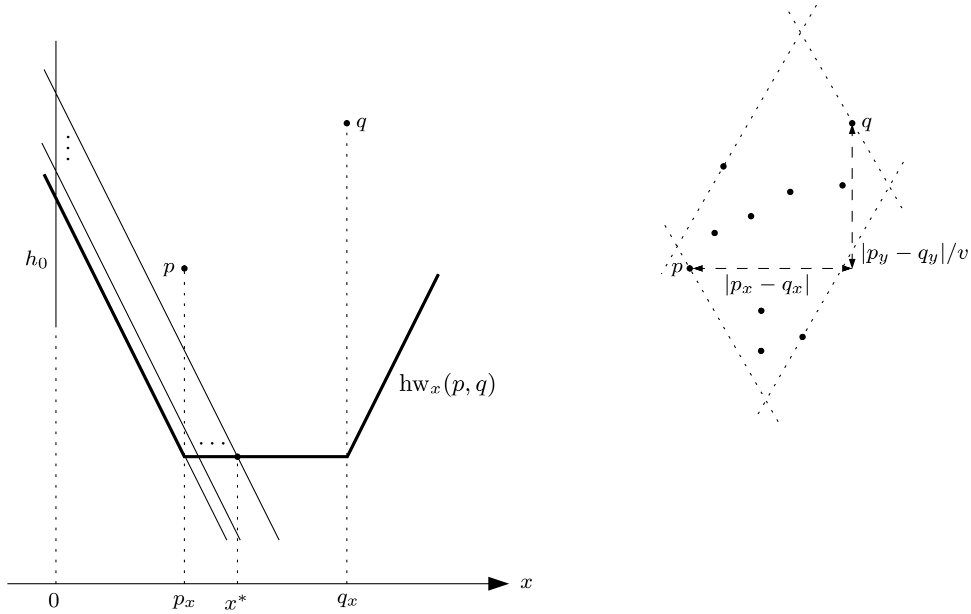

Figure 5: The highway graph of $\mathrm{hw}_x(p,q)$ has the highest valley floor. The pair $\{p,q\}$ is the diametral pair under the L_1 metric such that p is the point extreme in the direction of the vector $(-1,-v)$ and q is the point extreme in the direction of the vector $(1,v)$. The rightmost left ray intersects the highest valley floor at a point on $\mathcal{E}$, which is an optimal highway placement.

each pair $(p,q) \in S$, we can consider two sums of two functions,

$$f(p)+g(q) = (p_y - q_y)/v + p_x + q_x \text{ and } f(q)+g(p) = (q_y - p_y)/v + p_x + q_x.$$

From all pairs of points, the pair $\{p', q'\}$ with the maximum sum, $f(p')+g(q')$ or $f(q')+g(p')$, realizes the rightmost left ray. Note that this pair can be found by the two points extreme in the directions of the vectors $(1, \pm v)$. If $p' = q'$, then we find the second extreme points, p'' and q'' in each of directions and find the pair with maximum sum from $\{p', p''\}$ and $\{p', q''\}$.

Theorem 1 *Given n points in the plane, an optimal axis-parallel highway under the L_1 metric can be computed in linear time.*

3 The Optimal Highway under the Euclidean metric

Given a highway speed $v > 1$, let $\alpha = \sin^{-1}\frac{1}{v}$. For a point p, we denote by $\mathrm{cone}(p)$ the horizontal solid double cone with apex p and angle 2α in the plane. For a point p, we denote by p^+ the point on $h(x)$ with y-coordinate $p_y + |x - p_x| \tan\alpha$, and denote by p^- the point on $h(x)$ with y-coordinate $p_y - |x - p_x| \tan\alpha$.

$$f_x(p,q) = \begin{cases} (|p_x - x| + |q_x - x|)/\cos\alpha + (p_y^- - q_y^+)/v & \text{for } p_y \geqslant q_y, \\ (|p_x - x| + |q_x - x|)/\cos\alpha + (q_y^- - p_y^+)/v & \text{for } p_y < q_y. \end{cases}$$

We now define the highway distance of the pair $\{p, q\}$ as

$$\mathrm{hw}_x(p,q) = \begin{cases} d(p,q) & \text{if } q \in \mathrm{cone}(p), \\ f_x(p,q) & \text{if } q \notin \mathrm{cone}(p). \end{cases}$$

Then the travel time distance of the pair $\{p, q\}$ is defined as $t_{p,q}(x) = \min\{d(p, q), \mathrm{hw}_x(p, q)\}$.

Before we prove the main lemma, we need a technical lemma.

Lemma 4 *Let $\{p, q\}$ be a pair of points with $p_x \leqslant q_x$. Then, for x in $p_x \leqslant x \leqslant q_x$, $f_x(p, q) \leqslant d(p, q)$. add the reference to Sang Won's paper*

Now we are ready to give a lemma analogous to Lemma 2.

Lemma 5 *Let $\{a, b\}$ and $\{c, d\}$ be two pairs of points such that $\max\{a_x, b_x\} < \min\{c_x, d_x\}$, and $b \notin \mathrm{cone}(a)$ and $c \notin \mathrm{cone}(d)$. If $\mathrm{hw}_x(a, b) = \mathrm{hw}_x(c, d)$ for some $x = x_0$ in $\max\{a_x, b_x\} < x_0 < \min\{c_x, d_x\}$ then, there always exists a pair $\{\alpha, \beta\}$ such that $\alpha \in \{a, b\}$ and $\beta \in \{c, d\}$, and $\mathrm{hw}_{x_0}(\alpha, \beta) \geqslant \mathrm{hw}_{x_0}(a, b)$.*

Proof. Without loss of generality assume that $a_y > b_y$ and $c_y > d_y$. We also assume that $\min\{a_y^-, b_y^+\} \leqslant \min\{c_y^-, d_y^+\}$. Let $\Pi_{ab} = aa^-b^+b$ and $\Pi_{cd} = cc^-d^+d$ be two paths that connect a to b, and c to d via the highway $h(x_0)$, respectively.

Let $I = \Pi_{ab} \cap \Pi_{cd}$. There are two cases, either $I \neq \emptyset$ or $I = \emptyset$. For each of these two cases, we have a few subcases: (a) $a_y^- > b_y^+$ and $c_y^- > d_y^+$, (b) $a_y^- > b_y^+$ and $c_y^- < d_y^+$ (or its symmetric case, $a_y^- < b_y^+$ and $c_y^- > d_y^+$), and (c) $a_y^- < b_y^+$ and $c_y^- < d_y^+$ (see Figure 6 and Figure 7.) For each of these subcases, we define three values

$$\sigma_1 = (d_y^+ - b_y^+)/v, \qquad \sigma_2 = (b_y^+ - c_y^-)/v, \qquad \sigma_3 = (c_y^- - a_y^-)/v.$$

Then $f_{x_0}(b, c) = |bb^+| + |cc^-| - \sigma_2$ and $f_{x_0}(c, d) = |cc^-| + |dd^+| - \sigma_1 - \sigma_2$. If $f_{x_0}(b, c) \geqslant f_{x_0}(c, d)$, then we are done. Therefore, we assume that $f_{x_0}(b, c) < f_{x_0}(c, d)$. From this, we get $|bb^+| < |dd^+| - \sigma_1$.

$$\begin{aligned} \mathrm{hw}_x(a,b) = f_{x_0}(a,b) &= |aa^-| + |bb^+| - \sigma_2 - \sigma_3 \\ &< |aa^-| + |dd^+| - \sigma_1 - \sigma_2 - \sigma_3 \\ &\leqslant f_{x_0}(a,d) \\ &\leqslant \mathrm{hw}_{x_0}(a,d). \end{aligned}$$

(The last inequality follows from Lemma 4.) □

3.1 Algorithms

As we did under the L_1 metric, we now consider the highway time graphs $\mathrm{hw}_x(p, q)$ of all pairs $\{p, q\}$ of points in S. Then the function $\omega(x) = max_{\{p,q\} \subseteq S} \mathrm{hw}_x(p, q)$ represents their upper envelope $\mathcal{E}$. Clearly, the function $-\omega(x)$ is unimodal, and the x-interval of minimum travel time on $\mathcal{E}$ realizes the optimal highway placement. Again the x-interval of minimum travel time lies in the maximal interval of the highest valley floor.

Imagine that we have a vertical highway with speed $v > 1$ between every pair of points. Then the diametral pair can still be found by computing Farthest neighbour Voronoi diagram with the highway, which can be done in time $O(n \log n)$.

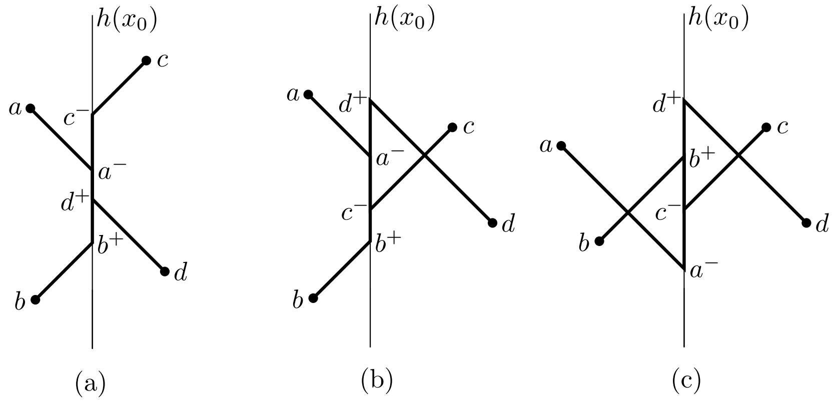

Figure 6: A tie with $\Pi_{ab} \cap \Pi_{cd} \neq \emptyset$.

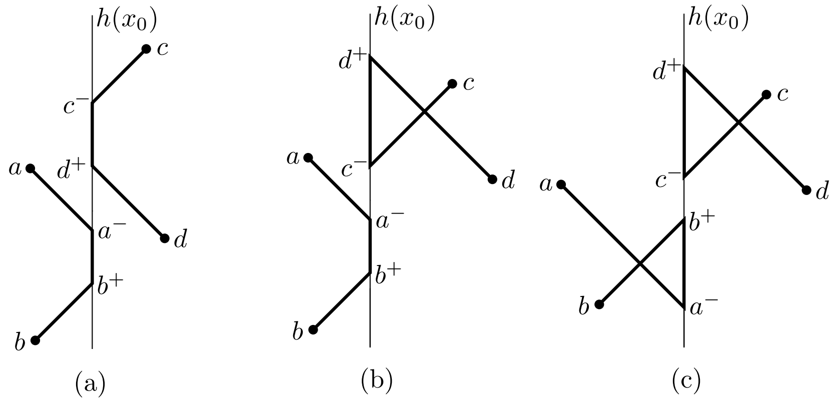

Figure 7: A tie with $\Pi_{ab} \cap \Pi_{cd} = \emptyset$.

Without loss of generality, assume now that all the points have positive x-coordinates, and the vertical highway lies at $x = 0$. Then we compute the left ray with maximum y-intercept from quadratic number of parallel rays, as we did for the L_1 metric. We may use the fast matrix searching technique for finding this ray. The intersection of this ray with the highest valley floor defines an optimal placement of the highway. *any other idea of doing it faster?*

Theorem 2 *Given n points in the plane, an optimal axis-parallel highway under the Euclidean metric can be computed in time $O(n \log n)$.*

4 Extensions

- What about placing the optimal combination of a vertical and a horizontal highway?

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