Let C be a circle of radius one, and let $0 < \delta < 1$. We wish to find the shortest closed curve H such that for any two points $p, q \in C$ with $d(p, q) > 2\delta$ we have $d(p, H) + d(q, H) \leq 2\delta$. As observed before, we can assume H to be convex.

For a convex set F in the plane, let $w(F, \theta)$ be the width of F in direction θ, that is, the length of the orthogonal projection of F onto a line with slope θ.

We make use of (a special case of) the Cauchy-Crofton formula, which asserts that the perimeter of F equals $\int_0^\pi w(F, \theta) d\theta$. We apply this to H, and learn that

$$\text{length}(H) = \int_0^\pi w(H, \theta) d\theta.$$

Consider now $0 \leq \theta < \pi$. H is enclosed in an infinite strip with slope $\theta + \pi/2$ and width $w(H, \theta)$. However, C contains two points p, q at distance 2, such that the line pq has slope θ. It follows that $d(p, H) + d(q, H) \geq 2 - w(H, \theta)$, which implies $2 - w(H, \theta) \leq 2\delta$, or $w(H, \theta) \geq 2 - 2\delta$. This gives

$$\text{length}(H) = \int_0^\pi w(H, \theta) d\theta \geq 2\pi(1 - \delta),$$

which is the circumference of a circle of radius $1 - \delta$. It follows that the conjectured solution is indeed optimal for all δ.

To prove that it is indeed the only possible solution, we need just a bit more work (there may be an easier way to do this). We first observe that since $w(H, \theta)$ is a continuous function of θ, in order to obtain an optimal highway we must have $w(H, \theta) = 2 - 2\delta$ everywhere.

Consider now again a diameter pq of C with slope θ. This diameter must intersect H in two points at distance $2 - 2\delta$ (otherwise, the width restriction stops us from fulfilling the distance requirement). In particular, this implies that the center of C lies inside H, and so we can parameterize H in polar coordinates as in Peter's writeup. The observation above shows that $r(\phi) + r(\phi + \pi) = 2 - 2\delta$, and as in Peter's writeup we see that the length is minimized when $r'(\phi) = 0$, and H is a circle.

Note that the same technique can be applied to non-circular cities C. Assume that C is a convex curve, and let $\delta > 0$. As above, we can observe that $w(H, \theta) \geq w(C, \theta) - 2\delta$, which implies that

$$\text{length}(H) \geq \text{length}(C) - 2\delta\pi.$$

Now, if C is smooth and its radius of curvature is everywhere larger than δ, then I believe it is known that tracing a point at distance δ away on the normal to the curve leads to a curve of exactly this length, showing that this offset curve is an optimal solution. I need to look for a reference, and haven't really thought about how to show that this is the *only* optimal solution.